Upper semi-continuity of the Royden-Kobayashi pseudo-norm, a counterexample for Hölderian almost complex structures.

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If X is an almost complex manifold, with an almost complex structure J of class $\mathcal{C}^\alpha$, for some $\alpha > 0$, for every point $p \in X$ and every tangent vector V at p, there exists a germ of J-holomorphic disc through p with this prescribed tangent vector. This existence result goes back to [N-W] (Theorem III.). See [I-R] (Appendix 1) for a re-writing and a generalization (to k-jets) of the proof. All the J holomorphic curves are of class $\mathcal{C}^{1,\alpha}$ ([N-W], [S]).

Then, exactly as for complex manifolds one can define the Royden-Kobayashi pseudo-norm of tangent vectors. The question arises whether this pseudo-norm is an upper semi-continuous function on the tangent bundle. For complex manifolds it is the crucial point in Royden's proof of the equivalence of the two standard definitions of the Kobayashi pseudo-metric ([R 1], [R 2], [L] pages 88-94). The upper semi-continuity of the Royden-Kobayashi pseudo-norm has been established by Kruglikov [K] for structures that are smooth enough. In [I-R], it is shown that $\mathcal{C}^{1,\alpha}$ regularity of J is enough.

Here we show the following:

Theorem 1. *There exists an almost complex structure J of class $\mathcal{C}^{\frac{1}{2}}$ on the unit bidisc $\mathbb{D}^2 \subset \mathbb{C}^2$, such that the Royden-Kobayashi pseudo-norm is not an upper semi-continuous function on the tangent bundle.*

The example is very explicit and very simple to describe. See Part II. We refer the reader unfamiliar with the above notions to [I-R]. For the proof of the failure of upper semi-continuity we shall need the following result:

Theorem 2. *For any continuous (complex valued) function f defined on the unit disc in $\mathbb{C}$, that is continuously differentiable on the set on which $f \neq 0$, and that on that set satisfies:*

$$\frac{\partial f}{\partial \overline{z}} \;=\; |f|^{\frac{1}{2}} \;,$$

one has $\sup_{|z|<1} |f(z)| \geq \frac{1}{10}$, if $f(0) \neq 0$.

So, roughly speaking, the theorem says that the equation $\frac{\partial f}{\partial \overline{z}} \;=\; |f|^{\frac{1}{2}}$, which has of course $f \equiv 0$ as a solution does not have small solutions with $f(0) \neq 0$. It can of course

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be compared with what happens for the ordinary differential equation $g' = |g|^{\frac{1}{2}}$ on the interval $[-1, +1]$. (Up to a factor 2, it is the equation to which $\frac{\partial f}{\partial \bar{z}} = |f|^{\frac{1}{2}}$ reduces if $f(z)$ depends only on the real part of z.) Every (real) solution g with $g(0) \neq 0$ satisfies $g(1) > \frac{1}{4}$ (if $g(0) > 0$), or $g(-1) < -\frac{1}{4}$ (if $g(0) < 0$). However, the equation has small non zero solutions that are identically 0 on large sub-intervals of $[-1, +1]$.

Part I.

The crucial step for proving Theorem 2 is the following Lemma.

Lemma 1. *Let ω be an open subset in $\mathbb{C}$. Let h be a continuously differentiable function defined on ω such that $\frac{\partial h}{\partial \bar{z}} = |h|^{\frac{1}{2}}$. If h does not vanish anywhere on ω, then h is smooth (C^∞) and*

$$\Delta\left(|h|^{\frac{3}{4}}\right) \geq \frac{3}{4}|h|^{-\frac{1}{4}}.$$

Proof. In the proof differentiation will be denoted by lower indices.

The smoothness of u follows from standard elliptic bootstrapping. Since the question is purely local, we can assume that ω is simply connected. Set $g = h^{\frac{1}{2}}$ (a determination of the square root of h). The hypothesis $h_{\bar{z}} = |h|^{\frac{1}{2}}$, is equivalent to $\left(h^{\frac{1}{2}}\right)_{\bar{z}} = \frac{1}{2}\left(\frac{\bar{h}}{h}\right)^{\frac{1}{4}}$. It can then be restated:

$$g_{\bar{z}} = \frac{1}{2}\left(\frac{\bar{g}}{g}\right)^{\frac{1}{2}}. \tag{1}$$

Set $g(z) = \rho e^{i\varphi(z)}$, with $\rho > 0$ and φ real valued. The conclusion to be reached is

$$\Delta(\rho^{\frac{3}{2}}) \geq \frac{3}{4}\rho^{-\frac{1}{2}}. \tag{$*$}$$

One has $g_{\bar{z}} = \rho_{\bar{z}} e^{i\varphi} + i\varphi_{\bar{z}} \rho e^{i\varphi}$, $\left(\frac{\bar{g}}{g}\right)^{\frac{1}{2}} = e^{-i\varphi}$. Hence (1) gives

$$\rho_{\bar{z}} + i\varphi_{\bar{z}}\rho = \frac{1}{2}e^{-2i\varphi}. \tag{2}$$

Separating real and imaginary parts:

$$\rho_x - \rho\varphi_y = \cos 2\varphi, \tag{3}$$

$$\rho_y + \rho\varphi_x = -\sin 2\varphi. \tag{3'}$$

Multiply the first equation by $-\varphi_y$, the second one by φ_x and add up in order to get:

$$\rho(\varphi_x^2 + \varphi_y^2) + (\rho_y\varphi_x - \rho_x\varphi_y) = -(\varphi_x \sin 2\varphi + \varphi_y \cos 2\varphi). \tag{4}$$

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Now, differentiate (3) with respect to x and (3') with respect to y and add up. One gets:

$$\Delta\rho + (\rho_y\varphi_x - \rho_x\varphi_y) = -2(\varphi_x\sin 2\varphi + \varphi_y\cos 2\varphi). \tag{5}$$

(4) and (5) yield:

$$\Delta\rho - 2\rho(\varphi_x^2 + \varphi_y^2) = \rho_y\varphi_x - \rho_x\varphi_y,$$

and therefore

$$2\rho\Delta\rho - 4\rho^2(\varphi_x^2 + \varphi_y^2) = 2\rho(\rho_y\varphi_x - \rho_x\varphi_y). \tag{6}$$

From equation (3) we have: $(\rho_x - \rho\varphi_y)^2 + (\rho_y + \rho\varphi_x)^2 = 1$, hence:

$$\rho_x^2 + \rho_y^2 + \rho^2(\varphi_x^2 + \varphi_y^2) + 2\rho(\rho_y\varphi_x - \rho_x\varphi_y) = 1. \tag{7}$$

From (6) and (7):

$$\rho_x^2 + \rho_y^2 + 2\rho\Delta\rho = 1 + 3\rho^2(\varphi_x^2 + \varphi_y^2) \geq 1. \tag{8}$$

The left hand side in (8) is equal to $\frac{4}{3}\rho^{\frac{1}{2}}\Delta\bigl(\rho^{\frac{3}{2}}\bigr)$. This establishes (*).

Q.E.D.

Lemma 2. *Let u be a non-negative subharmonic function on the unit disc in $\mathbb{C}$. Assume that on the set on which $u \neq 0$, one has $\Delta u \geq 1$. If $u(0) > 0$, then $\sup_{|z|<1} u(z) > \frac{1}{4}$.*

Proof. Set $\omega = \{z \in \mathbb{D};\ u(z) \neq 0\}$. The function $v = u - \frac{1}{4}(x^2 + y^2)$ is a subharmonic function on ω. If $\mathrm{Sup}_{|z|<1} u(z) \leq \frac{1}{4}$, on the whole boundary of ω we would have $v \leq 0$, which is impossible by the maximum principle since $v(0) = u(0) > 0$.

Q.E.D.

Proof of Theorem 2. Assume that $\sup|f| < \frac{1}{10}$. Then Lemma 1 gives $\Delta\bigl(|f|^{\frac{3}{4}}\bigr) \geq \frac{3}{4}10^{\frac{1}{4}} \geq 1$ on ω. So we can apply Lemma 2, and one therefore gets $\sup_{|z|<1}|f(z)|^{\frac{3}{4}} \geq \frac{1}{4}$, and therefore $\sup_{|z|<1}|f(z)| \geq \frac{1}{10}$, a contradiction, as desired.

Q.E.D.

Part II.

For $r > 0$, $\mathbb{D}_r$ will denote the open disc $\{z = (x+iy) \in \mathbb{C};\ |z| < r\}$. For the unit disc $\mathbb{D}_1$ in $\mathbb{C}$, we will abbreviate to $\mathbb{D}$. We will identify $\mathbb{R}^4$, with coordinates (x_1, y_1, x_2, y_2) with $\mathbb{C}^2$ (setting $z_j = (x_j + iy_j)$). Let $\Omega = \mathbb{D}_2 \times \mathbb{D}_{\frac{1}{10}}$. On Ω consider the almost complex structure J defined by:

$$J = J(x_1, y_1, x_2, y_2) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -1 \\ \lambda & 0 & 1 & 0 \end{pmatrix}$$

where $\lambda(x_1, y_1, x_2, y_2) = -2(x_2^2 + y_2^2)^{\frac{1}{4}} = -2|z_2|^{\frac{1}{2}}$. Note that $J^2 = -\mathbf{1}$.

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In the next lines, we use complex notations. The map from $\mathbb{D}_2$ into Ω: $z \mapsto (z, 0)$ is a J-holomorphic map. This shows that the Royden-Kobayashi pseudo-norm of the vector $(1, 0)$ tangent at the point $(0, 0)$, is $\leq \frac{1}{2}$.

CLAIM: *There exists $A < 2$ such that if Z is a J-holomorphic map from $\mathbb{D}_r$ into Ω with:*

$$Z(0) = (0, b) \text{ with } b \neq 0 \text{ , and } \frac{\partial Z}{\partial x}(0) = (1, 0) \text{ ,}$$

then $r \leq A$.

The claim implies that the vector $(1, 0)$ tangent at the point $(0, b)$ has Royden-Kobayashi pseudo-norm $\geq \frac{1}{A} > \frac{1}{2}$. So, by identification of Ω and $\mathbb{D}^2$, to prove Theorem 1, it is enough to check the claim.

Proof of the Claim. To say that a map $z \mapsto Z(z) = (Z_1(z), Z_2(z))$, from $\mathbb{D}_r$ to Ω is J-holomorphic, means that

$$\frac{\partial Z}{\partial y}(z) = J(Z(z)) \frac{\partial Z}{\partial x}(z).$$

It immediately gives us that Z_1 must be a holomorphic function of z. By the Schwarz Lemma, if r is close enough to 2, and $Z_1(0) = 0$, $Z_1'(0) = 1$, Z_1 must be close to the identity, say on the disc of radius $\frac{3}{2}$. More precisely, there exists $A < 2$ so that if $r > A$ there is a holomorphic map φ defined on the unit disc, close to the identity, with $\varphi(0) = 0$ and $\varphi'(0) = 1$ such that $Z_1(\varphi(z)) \equiv z$. Then consider the J-holomorphic map from $\mathbb{D}$ into Ω obtained by re-parameterization:

$$z \mapsto Z^{\#}(z) = Z(\varphi(z)) = (z, f(z)).$$

In order to prove the claim, we need to show that such a J-holomorphic map $Z^{\#}$ does not exist with $f(0) \neq 0$.

We now look at the condition $\frac{\partial Z^{\#}}{\partial y}(z) = J(Z^{\#}(z)) \frac{\partial Z^{\#}}{\partial x}(z)$, which is the condition for $Z^{\#}$ to be J-holomorphic. Set $f = u + iv$. Looking at the (x_2, y_2) coordinates, one gets:

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

$$\frac{\partial v}{\partial y} = \lambda + \frac{\partial u}{\partial x}.$$

Multiply the first equation by i and subtract from the second equation. One gets $\frac{\partial(u+iv)}{\partial \overline{z}} = -\frac{\lambda}{2}$.

So the condition for the map $Z^{\#}$ be J-holomorphic, with J as above, is simply that $\frac{\partial f}{\partial \overline{z}} = |f(z)|^{\frac{1}{2}}$. If $f(0) \neq 0$, Theorem 2 implies that $\text{Sup}_{|z|<1}|f(z)| > \frac{1}{10}$, which is impossible since $Z^{\#}$ maps $\mathbb{D}$ into Ω.

Comments.

1) It is well known that for rough (non $\mathcal{C}^1$) almost complex structures there is no uniqueness result: Two J-holomorphic maps defined on $\mathbb{D}$ can coincide on a non empty open set without being identical. It indeed happens in the above example. With the above notations, taking f depending only on x, it reduces to the non-uniqueness property for the O.D.E. $g' = |g|^{\frac{1}{2}}$. An example of J-holomorphic map $Z^{\#}$ (defined near 0) is then given by taking $f(z) = 0$ if $x \leq 0$, and $f(z) = x^2$ for $x > 0$. A strong relation seems to exist between the lack of uniqueness for O.D.E. and the lack of 'small solutions'.

2) A minor modification of the proof leads to an example with an almost complex structure Hölderian of Hölder exponent $\frac{2}{3}$, instead of only $\frac{1}{2}$.

3) Interestingly, there is another situation somewhat comparable to the situation in this paper, and to which L. Lempert drew the attention of one of us a few years ago. In [C-R] there is a very simple construction of holomorphic discs, depending on a parameter, simply using the implicit function Theorem, in complex dimension 2. L. Lempert pointed out that, in higher dimensions, one could still prove the existence of similar discs, by applying the Schauder fixed point theorem. But it did not give the dependence on parameters as needed in Kontiuitätsatz arguments. It could not due to the failure of the Hartogs-Chirka Theorem in complex dimension > 2 ([Ro]). For J-holomorphic curves, proving the existence of curves with prescribed tangent can be done with the implicit function Theorem if J is of class $\mathcal{C}^{1,\alpha}$. But for $\mathcal{C}^\alpha$ regularity of J, the proof as in [N-W] uses the Schauder fixed point Theorem. The non upper semi-continuity of the Royden-Kobayashi pseudo-norm comes from the impossibility of small perturbations.

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